$$A = \frac{\frac{3}{2}\int_0^{2\pi}\int_0^{\pi} n_1(w_{1,x})^2\,dx\,d\tau}{\Omega_0\int_0^{2\pi}\int_0^{\pi}(\dot{w}_1)^2\,dx\,d\tau} = 0 \tag{40}$$

and B is determined to be

$$B = \frac{2\int_0^{2\pi}\int_0^{\pi} n_1 w_{1,x} w_{2,x}\,dx\,d\tau + \int_0^{2\pi}\int_0^{\pi} n_2(w_{1,x})^2\,dx\,d\tau}{\Omega_0\int_0^{2\pi}\int_0^{\pi}(\dot{w}_1)^2\,dx\,d\tau} \tag{41}$$

$$= (3\pi/16\Omega_0)[1-(5\pi r^2/2\Omega_0)]$$

Consequently

$$\Omega/\Omega_0 = 1+(3\pi/16\Omega_0)[1-(5\pi r^2/2\Omega_0)]\xi^2+\cdots \tag{42}$$

Initially as r increases, the behavior trend is one of softening (decreasing frequency). This persists until

$$r^2 = 3/\pi \qquad B_{min} = -3\pi/20 \tag{43}$$

Thereafter a reversed hardening trend appears that approaches the neutral limit $B = 0$ as r^2 becomes large.

Conclusions

It has been found that as the rise of an arch increases, the free vibration behavior in the fundamental mode first exhibits a softening trend due to curvature. This trend is reversed as the rise parameter exceeds the value given in Eq. (43). A neutral limit is ultimately approached for large values of rise.

References

[1] Von Federhofer, K., "Nicht-lineare Biegungsschwingungen des Kreisringes," *Ingenieur-Archive*, Bd. XXVIII, 1959, pp. 53–58.

[2] Evensen, D. A., "Theoretical and Experimental Study of the Nonlinear Flexural Vibrations of Thin Circular Rings," TR R-227, Dec. 1965, NASA.

[3] Chu, H. N., "Influence of Large Amplitudes on Flexural Vibrations of a Thin Circular Cylindrical Shell," *Journal of Aerospace Sciences*, Vol. 28, No. 8, 1961, pp. 602–609.

[4] Evensen, D. A. and Fulton, R. E., "Some Studies on the Nonlinear Dynamic Response of Shell-Type Structures," *Dynamic Stability of Structures*, edited by G. Herrmann, Pergamon Press, New York, 1967, pp. 237–254.

[5] Mayers, J. and Wrenn, B. G., "On the Nonlinear Free Vibrations of Thin Circular Cylindrical Shells," SUDAAR 269, 1966, Stanford Univ., Stanford, Calif.

[6] Olson, M. D., "Some Experimental Observations on the Nonlinear Vibrations of Cylindrical Shells," *AIAA Journal*, Vol. 3, No. 9, Sept. 1965, pp. 1775–1777.

[7] Ginsberg, J. H., "Nonlinear Resonant Vibrations of Infinitely Long Cylindrical Shells," *AIAA Journal*, Vol. 10, No. 8, Aug. 1972, pp. 979–980.

[8] Rehfield, L. W., "Nonlinear Free Vibrations of Elastic Structures," *International Journal of Solids and Structures*, Vol. 9, No. 5, May 1973, pp. 581–590.

[9] Koiter, W. T., "On the Stability of Elastic Equilibrium," TT F-10,833, 1967, NASA.

[10] Koiter, W. T., "Elastic Stability and Post-Buckling Behavior," *Nonlinear Problems*, edited by R. E. Langer, Univ. of Wisconsin Press, Madison, Wis., 1963, pp. 257–275.

[11] Budiansky, B., "Dynamic Buckling of Elastic Structures: Criteria and Estimates," *Dynamic Stability of Structures*, edited by G. Herrmann, Pergamon Press, New York, 1966, pp. 83–106.

[12] Fung, Y. C. and Kaplan, A., "Buckling of Low Arches or Beams of Small Curvature," TN 2840, 1952, NACA.

[13] Hoff, N. J. and Bruce, V. G., "Dynamic Analysis of the Buckling of Laterally Loaded Flat Arches," *Journal of Mathematics and Physics*, Vol. 32, No. 4, Jan. 1954, pp. 276–288.

[14] Hsu, C. S., "Effects of Various Parameters on the Dynamic Stability of a Shallow Arch," *Journal of Applied Mechanics*, Vol. 34, No. 2, June 1967, pp. 349–358.

[15] Hsu, C. S., "Stability of Shallow Arches Against Snap-Through Under Timewise Step Loads," *Journal of Applied Mechanics*, Vol. 35, No. 1, March 1968, pp. 31–39.

[16] Hsu, C. S., Kuo, C. T., and Lee, S. S., "On the Final States of Shallow Arches on Elastic Foundations Subjected to Dynamical Loads," *Journal of Applied Mechanics*, Vol. 35, No. 4, Dec. 1968, pp. 713–723.

Laminar Boundary-Layer Response to Freestream Disturbances

P. J. Erens* and V. A. L. Chasteau†
University of Pretoria, Pretoria, South Africa

Nomenclature

f = oscillation frequency (Hz)
F = dimensionless frequency parameter $= 2\pi f v/U_\infty^2$
Re_{δ_1} = Reynolds number based on displacement thickness $= U\delta_1/v$
Re_x = Reynolds number based on distance from leading edge
u' = instantaneous velocity fluctuation about mean velocity (m/sec)
U_∞ = freestream velocity (m/sec)
x = distance from leading edge of plate (m)
v = kinematic viscosity (m²/sec)
δ = boundary-layer thickness (m)
δ_1 = displacement thickness $= 1{,}721\,(vx/U_\infty)^{1/2}$, (m)

During the course of an experimental investigation into the effect of two-dimensional surface roughnesses on laminar boundary-layer stability[1] some measurements were made of the growth and decay of the amplitude of naturally occurring oscillations in a laminar boundary layer. These measurements were made along a flat plate in air with zero pressure gradient.

The method of measurement consisted of a determination of the frequency spectrum of the streamwise component of the laminar boundary-layer velocity fluctuations, u', at various points along the length of the plate. In order to do this the linearized output signal from a hot-wire anemometer, located in the boundary layer, was passed through a frequency analyzer. By observing the behavior of the boundary-layer oscillations within a narrow frequency band as the hot-wire was moved downstream, the amplitudes of the disturbance around the particular center frequency could be plotted as a function of distance along the plate.

Center frequencies in the range of 100–350 Hz with a 4-Hz bandwidth were studied. The hot-wire was mounted at a height of 20% of the local boundary-layer thickness, δ, above the surface. It was found that this height was not critical as long as the same proportional depth was adhered to in all cases. The output signal from the analyzer was integrated and averaged over a period of 20 sec for each reading. The freestream

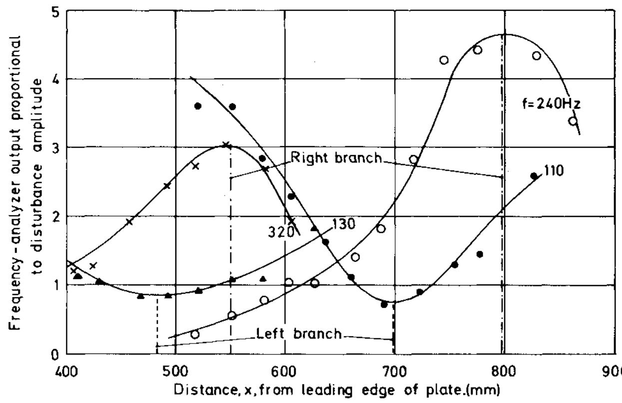

Fig. 1 Growth and decay of selected disturbances in the boundary layer. Freestream velocity U_∞ = 20 m/sec.

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* Lecturer, Department of Mechanical Engineering.
† Senior Lecturer, Department of Mechanical Engineering.

velocity, U_∞, was held constant at 20 m/sec for most of the measurements although a few readings were also taken at 16 m/sec to extend the range of the parameters. The freestream turbulence level, $(u'^2)^{1/2}/U_\infty$, was 0.4% and transition to fully turbulent flow as measured by a boundary-layer pitot-tube took place at a Reynolds number $Re_x = 1.1 \times 10^6$. In the region that the measurements were made the amplitudes of the oscillations measured at selected center frequencies were at least ten times greater than those in the freestream.

Some typical results obtained are illustrated in Fig. 1. Points of maximum and minimum velocity fluctuation amplitude (denoted right and left branch, respectively) were evident. At the distances, x, downstream of the leading edge at which these maxima and minima occurred, the Reynolds numbers, Re_{δ_1}, were calculated using the theoretical laminar boundary-layer displacement thickness, δ_1, at the point x. The center frequencies, f, were converted to a nondimensional frequency parameter $F = 2\pi f\nu/U_\infty{}^2$ where ν was the kinematic viscosity. Values of F were plotted against Re_{δ_1}, as shown in Fig. 2.

It is interesting to compare these results to those determined experimentally and theoretically for the stability of sinusoidal disturbances in a laminar boundary layer. Previous researchers[2,3] have used a vibrating ribbon to introduce sinusoidal disturbances into the boundary layer and a hot-wire anemometer to determine their subsequent growth or decay in wind tunnels with extremely low turbulence levels (less than 0.08%). In these studies the initial values, determined by the vibrating ribbon, and the boundary conditions, determined by the low level of turbulence, are fixed. The stability of the boundary layer can then be studied according to methods discussed by Schlichting et al.[4-6]

In the present tests energy entered the boundary layer from the freestream all the way up to the points of measurement, and the boundary-layer oscillations measured were the end result of the boundary layer response to the freestream disturbances. It is possible that some disturbances were created by slight roughnesses on the surface used, as this consisted of a hard rubber sheet sprayed with lacquer; however, the results compared well with those on a smooth perspex plate.

In some further studies[1] the surface was deflected by drawing it onto a series of regularly spaced rods, transverse to the direction of flow, with a vacuum. This gave the effect of a wavy surface with waves at right angles to the flow direction. In one instance, waves of 14.5 mm wavelength and crest to trough amplitudes of 0.12 mm and 0.22 mm with freestream velocity of 20 m/sec were used. Similar response measurements were taken and are shown in Fig. 3. Further tests with a wavelength of 57.5 mm produced extremely rapid transition to turbulent flow. With a freestream velocity of 20 m/sec a very dominant disturbance frequency of 370 Hz was created. This frequency corresponded to a dimensionless frequency parameter of $F = 96.5$ and transition occurred at $R_{\delta_1} = 1160$.

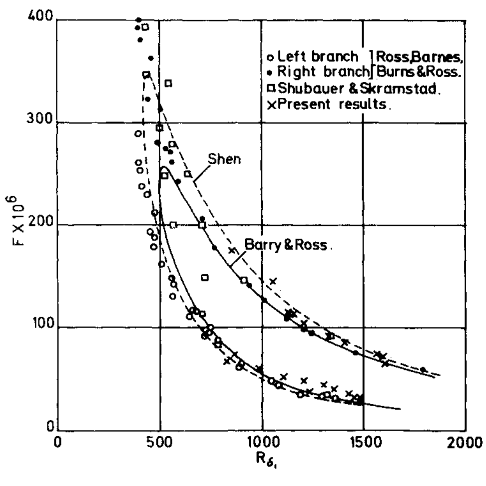

Fig. 2 Comparison of theoretical stability curves of Shen (1954) and Barry and Ross (1970) with experimental results.

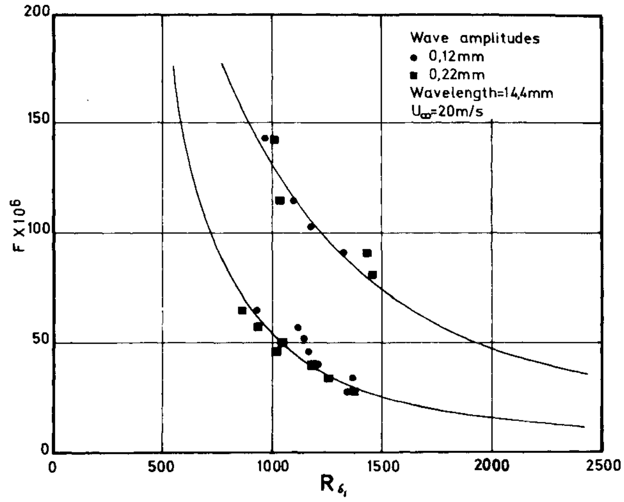

Fig. 3 Comparison of experimental points using wavy surface with curve due to Barry and Ross (1970).

In comparing the theoretical and experimental results for laminar boundary-layer stability and the present response results, it appears that, in the region considered, the inherent stability properties of the boundary layer dominate its response to disturbances originating from freestream turbulence and from surface irregularities. The naturally occurring oscillations represented disturbances which must have undergone considerable amplification in the boundary layer in the first instance since they were more than about ten times as great as the freestream velocity fluctuations (u') in the region of interest. In the neighborhood of the theoretical stability envelope, the various frequency components of these strong naturally occurring oscillations, with an, at present largely undefined, far upstream history, appeared to be influenced in the same manner as ideal sinusoidal disturbances under controlled conditions.

Klebanoff and Tidstrom[7] have recently studied the effect of a single two-dimensional roughness element (in the form of a cylindrical tripwire) on the stability of a laminar boundary layer by examining disturbance spectra. It appears that downstream of such an element the flow has a destabilizing influence on existing disturbances although this effect is probably very much dependent on the shape of the roughness element. In the present studies regular rounded surface waves were used as compared to the rather abrupt roughness elements used by Klebanoff and Tidstrom.

References

[1] Erens, P. J., "An Experimental Study of the Effect of a Surface with Transverse Waves on the Stability of a Laminar Boundary-Layer," M.S. dissertation, 1972, Univ. of Pretoria, Pretoria, South Africa.

[2] Schubauer, G. B. and Skramstad, H. K., "Laminar Boundary-Layer Oscillations and Transition on a Flat Plate," Rept. 909, 1950, NACA.

[3] Ross, J. A., Barnes, F. H., Burns, J. G., and Ross, M. A. S., "The Flat Plate Boundary Layer: Part 3, Comparison of Theory with Experiment," *Journal of Fluid Mechanics*, Vol. 43, Pt. 4, 1970, pp. 819–832.

[4] Schlichting, H., *Boundary-Layer Theory*, 6th ed., McGraw-Hill, New York, 1966.

[5] Lin, C. C., *The Theory of Hydrodynamic Stability*, Cambridge University Press, Cambridge, England, 1955.

[6] Shen, S. F., "Calculated Amplified Oscillations in the Plane Poiseuille and Blasius Flows," *Journal of the Aeronautical Sciences*, Vol. 24, 1954, pp. 62–64.

[7] Klebanoff, P. S. and Tidstrom, K. D., "Mechanism by Which a Two-Dimensional Roughness Element Induces Boundary-Layer Transition," *The Physics of Fluids*, Vol. 15, 1972, pp. 1173–1188.